Non-Autonomous Hamiltonian Systems and Morales-Ramis Theory

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Abstract

In this paper we present an approach towards the comprehensive analysis of the non-integrability of differential equations in the form $\ddot{x} = f(x,t)$ which is analogous to Hamiltonian systems with $1 + 1/2$ degree of freedom. In particular, we analyze the non-integrability of some important families of differential equations such as Painlevé II, Sitnikov and Hill-Schrödinger equation. We emphasize in Painlevé II, showing its non-integrability through three different Hamiltonian systems, and also in Sitnikov in which two different version including numerical results are shown. The main tool to study the non-integrability of these kind of Hamiltonian systems is Morales-Ramis theory. This paper is a very slight improvement of the talk with the same title delivered by the author in SIAM Conference on Applications of Dynamical Systems 2007.

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1 Introduction

In this section we present the necessary theoretical background to understand the rest of the paper.

1.1 Differential Galois Theory

Our theoretical framework consists of a well-established crossroads of Dynamical Systems theory, Algebraic Geometry and Differential Algebra. See [20] or [34] for further information and details. Given a linear differential system with coefficients in $\mathbb{C}(t)$,

$$\dot{\boldsymbol{z}} = A\,(t)\,\boldsymbol{z}, \tag{1}$$

a differential field $L \supset \mathbb{C}(t)$ exists, unique up to $\mathbb{C}(t)$-isomorphism, which contains all entries of a fundamental matrix $\Psi = [\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_n]$ of (1). Moreover, the group of automorphisms of this field extension, called the *differential Galois group* of (1), is an algebraic group G acting over the $\mathbb{C}$-vector space $\langle \boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_n \rangle$ of solutions of (1) and containing the monodromy group of (1).

It is worth recalling that the integrability of a linear system (1) is equivalent to the solvability of the identity component G^0 of the differential Galois group G of (1) – in other words, equivalent to the *virtual solvability* of G.

It is well established (e.g. [21]) that any linear differential equation system with coefficients in a differential field K

$$\frac{d}{dt}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \tag{2}$$

by means of an elimination process, is equivalent to the second-order equation

$$\ddot{\xi} - \left(a(t) + d(t) + \frac{\dot{b}(t)}{b(t)}\right)\dot{\xi} - \left(\dot{a}(t) + b(t)c(t) - a(t)d(t) - \frac{a(t)\dot{b}(t)}{b(t)}\right)\xi = 0, \tag{3}$$

where $\xi := \xi_1$. Furthermore, any equation of the form $\ddot{z} - 2p\dot{z} - qz = 0$, can be transformed, through the change of variables $z = ye^{\int p}$, into $\ddot{y} = -ry$, r satisfying the Riccati equation $\dot{p} = r + q + p^2$. This change is useful since it restricts the study of the Galois group of $\ddot{y} = -ry$ to that of the algebraic subgroups of $\mathrm{SL}_2\,(\mathbb{C})$.

A natural question which now arises is to determine what happens if the coefficients of the differential equation are not all rational. A new method was developed in [3], in order to transform a linear differential equation of the form $\ddot{x} = r(t)x$, with transcendental or algebraic non-rational coefficients, into its algebraic form – that is, into a differential equation with rational coefficients. This is called the *algebrization method* and is based on the concept of *Hamiltonian change of variables* [3]. Such a change is derived from the solution of a one-degree-of-freedom classical Hamiltonian.

Definition 1.1 (Hamiltonian change of variables) *A change of variables $\tau = \tau\,(t)$ is called **Hamiltonian** if $(\tau(t), \dot{\tau}\,(t))$ is a solution curve of the autonomous Hamiltonian system X_H with Hamiltonian function*

$$H = H(\tau, p) = \frac{p^2}{2} + \widehat{V}(\tau), \text{ for some } \widehat{V} \in \mathbb{C}(\tau).$$

Theorem 1.2 (Acosta-Blázquez algebrization method [3]) *Equation $\ddot{x} = r(t)x$ is algebrizable by means of a Hamiltonian change of variables $\tau = \tau(t)$ if, and only if, there exist f, α such that $\frac{d}{d\tau}(\ln \alpha), \frac{f}{\alpha} \in \mathbb{C}(\tau)$, where*

$$f(\tau(t)) = r(t), \quad \alpha(\tau) = 2(H - \widehat{V}(\tau)) = (\dot{\tau})^2.$$

Furthermore, the algebraic form of $\ddot{x} = r(t)x$ is

$$\frac{d^2x}{d\tau^2} + \left(\frac{1}{2}\frac{d}{d\tau}\ln\alpha\right)\frac{dx}{d\tau} - \left(\frac{f}{\alpha}\right)x = 0. \quad \square \tag{4}$$

The next intended step, once a differential equation has been algebrized, is studying its Galois group and, as a causal consequence, its integrability. Concerning the latter, and in virtue of the invariance of the identity component of the Galois group by finite branched coverings of the independent variable (Morales-Ruiz and Ramis, [23, Theorem 5]), it was proven in [3, Proposition 1] that the identity component of *the Galois group is preserved in the algebrization mechanism.*

The final step is analyzing the behavior of $t = \infty$ (or $\tau = \infty$) by studying the behavior of $\eta = 0$ through the change of variables $\eta = 1/t$ (or $\eta = 1/\tau$) in the transformed differential equation, i.e. $t = \infty$ (or $\tau = \infty$) is an ordinary point (resp. a regular singular point, an irregular singular point) of the original differential equation if, and only if, $\eta = 0$ is one such point for the transformed differential equation.

1.2 Morales-Ramis Theory

Everything is considered in the complex analytical setting from now on. The heuristics of the titular theory rest on the following general principle: if we assume system

$$\dot{\boldsymbol{y}} = X(\boldsymbol{z}) \tag{5}$$

"integrable" in some reasonable sense, then the corresponding variational equations along any integral curve $\Gamma = \{\widehat{\boldsymbol{z}}(t) : t \in I\}$ of (5), defined in the usual manner

$$\dot{\boldsymbol{\xi}} = X'(\widehat{\boldsymbol{z}}(t))\,\boldsymbol{\xi}, \tag{VE$_\Gamma$}$$

must be also integrable – in the Galoisian sense of the last paragraph in 1.1. We assume Γ, a Riemann surface, may be locally parametrized in a disc I of the complex plane; we may now complete Γ to a new Riemann surface $\overline{\Gamma}$, as detailed in [23, §2.1] (see also [20, §2.3]), by adding equilibrium points, singularities of the vector field and possible points at infinity.

The aforementioned "reasonable" sense in which to define integrability if system (5) is *Hamiltonian* is obviously the one given by the Liouville-Arnold Theorem, and thus the above general principle does have an implementation:

Theorem 1.3 (J. Morales-Ruiz & J.-P. Ramis, 2001) *Let H be an n-degree-of-freedom Hamiltonian having n independent first integrals in pairwise involution, defined on a neighborhood of an integral curve $\overline{\Gamma}$. Then, the identity component $\mathrm{Gal}(\mathrm{VE}_{\overline{\Gamma}})^0$ is an abelian group (i.e. $\mathrm{Gal}(\mathrm{VE}_{\overline{\Gamma}})$ is* virtually abelian*).*

See [23, Corollary 8] or [20, Theorem 4.1] for a precise statement and a proof.

1.3 Non Autonomous Hamiltonian Systems

Non-autonomous Hamiltonian systems (*NAHS*) on symplectic manifolds have long been the subject of study, and appear in a most natural way in Classical Mechanics and Control Theory, e.g. [1], [5], [17], [19], [18], [26], [31], [32].

We consider *NAHS* of the form

$$H = H(q_1, p_1, t) = \frac{p_1^2}{2} + V(q_1, p_1, t), \tag{6}$$

H is an *NAHS* with $1 + 1/2$ degree of freedom. It is well-known (e.g. [26]) that (6) can be included as a subsystem of the Hamiltonian system with two degree of freedom given by

$$\widehat{H} = \widehat{H}(q_1, q_2, p_1, p_2) = \frac{p_1^2}{2} + V(q_1, p_1, q_2) + p_2, \tag{7}$$

where q_2 and p_2 are conjugated variables, i.e. $p_2 = -H + k$, where k is constant, and $q_2 = t$. Furthermore, p_2 is easily seen to be the offset or counterbalancing energy of the system ([26], [28]).

Also worth mentioning are some recent results on canonical transformations in the extended phase space [28], [29], [30], [33], as well as on definitions and consequences of "integrability" under such circumstances or generalizations thereof, even for non-Hamiltonian systems ([10], [9], [12], [16]) which we will not delve into further at this point.

2 Main results

Consider the differential equation

$$\ddot{x} = f(x, t), \tag{8}$$

with particular solution $x = x(t)$. We will henceforth order our choice of positions as $q_1 = x$ and $q_2 = t$, thus yielding a Hamiltonian system given by

$$H = \frac{p_1^2}{2} - F(q_1, q_2), \quad F_{q_1}(q_1, q_2) = \frac{\partial F(q_1, q_2)}{\partial q_1} = f(q_1, q_2).$$

Equation (8) is obviously equivalent to Hamilton's equations for H,

$$\dot{q}_1 = p_1 = H_{p_1} \qquad \dot{p}_1 = -H_{q_1} = f(q_1, q_2);$$

this *NAHS* is included as a subsystem of $X_{\widehat{H}}$ linked to $\widehat{H} := H + p_2$, such as in equation (7). Assuming $x(t) = q_1(t)$ to be a solution of (8) and $q_2(t) = t$, we obtain an integral curve $\Gamma = \{\boldsymbol{z}(t)\}$ of $\widehat{H}$, where

$$\boldsymbol{z}(t) := (q_1(t), q_2(t), p_1(t), p_2(t)) = (q_1(t), t, \dot{q}_1(t), -H(t)).$$

We may now introduce our first main result:

Theorem 2.1 *Let Γ be an integral curve of $X_{\widehat{H}}$ such as the one introduced above. If $X_{\widehat{H}}$ is integrable by means of rational or meromorphic first integrals, then the Galois group of*

$$\ddot{\xi} = (f_{q_1}(q_1, q_2)|_\Gamma)\, \xi, \tag{9}$$

is virtually abelian.

Proof. The Hamiltonian field $X_{\widehat{H}}$ is given by $X_{\widehat{H}} = (p_1, 1, f(q_1, q_2), F_{q_2}(q_1, q_2))^T$. The variational equation VE_Γ along Γ is

$$\frac{d}{dt}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} = \left.\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ f_{q_1}(q_1, q_2) & f_{q_2}(q_1, q_2) & 0 & 0 \\ F_{q_1 q_2}(q_1, q_2) & F_{q_2 q_2}(q_1, q_2) & 0 & 0 \end{pmatrix}\right|_\Gamma \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix}, \tag{10}$$

More precisely, equation (10) is

$$\left\{\begin{array}{lcl} \dot{\xi}_1 & = & \xi_3, \\ \dot{\xi}_2 & = & 0, \\ \dot{\xi}_3 & = & (f_{q_1}(q_1, q_2)|_\Gamma)\, \xi_1 + (f_{q_2}(q_1, q_2)|_\Gamma)\, \xi_2, \\ \dot{\xi}_4 & = & (F_{q_1 q_2}(q_1, q_2)|_\Gamma)\, \xi_1 + (F_{q_2 q_2}(q_1, q_2)|_\Gamma)\, \xi_2. \end{array}\right. \tag{11}$$

Hence $\xi_2 = k$, where k is constant. Assuming $k = 0$, the *normal variational equations* (NVE_Γ, see [8, §1], [23, §4.3], [20, §4.1.3]) for $\widehat{H}$ are given by

$$\frac{d}{dt}\begin{pmatrix} \xi_1 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ f_{q_1}(q_1, q_2)|_\Gamma & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_3 \end{pmatrix}, \tag{12}$$

and solving (12) we can obtain ξ_4. System (12) is equivalent to equation (9), where $\xi = \xi_1$. In virtue of [20, Proposition 4.2], the virtual commutativity of $\mathrm{Gal}\,(\mathrm{VE}_\Gamma)$ implies that of $\mathrm{Gal}\,(\mathrm{NVE}_\Gamma)$; this, coupled with Theorem 1.3, implies that if $X_{\widehat{H}}$ is rationally or meromorphically integrable, then the Galois group of the equation (9) is virtually abelian. ■

Remark 1 The above disjunctive between *meromorphic* and *rational* Hamiltonian integrability is related to the status of $t = \infty$ as a singularity for the normal variational equations. More specifically, and besides the non-abelian character of the identity component of the Galois group, in order to obtain Galoisian obstructions to the *meromorphic* integrability of $\widehat{H}$ the point at infinity must be a regular singular point of (9). On the other hand, for there to be an obstruction to complete sets of *rational* first integrals, $t = \infty$ must be a irregular singular point.

Corollary 2.2 *If the Galois group of the differential equation $\ddot{\xi} = k(t)\xi$, is not virtually abelian, then, defining $H := \frac{p_1^2}{2} - k(q_2)\frac{q_1^2}{2}$ and $\widehat{H} := H + p_2$, $X_{\widehat{H}}$ is not integrable by means of meromorphic or rational first integrals.* □

Remark 2 If the Galois group of the equation $\ddot{\xi} = k(t)\xi$ is the Borel group $G = \mathbb{C}^* \ltimes \mathbb{C}$ (hence connected, solvable and non-abelian), $X_{\widehat{H}}$ is neither meromorphically nor rationally integrable, although it is still possible to solve the equation – as well as, ostensibly, the *NAHS* X_H.

In expectance of the following Corollary consider, for any $g(x)$, $a(t)$ and $\alpha = \alpha_0$ given, the equation

$$\ddot{x} = g_x(x)(g(x) + a(t)) + \alpha, \quad \alpha \in \mathbb{C}, \tag{13}$$

having a certain known particular solution $q_1 = q_1(t)$. Let

$$H = \frac{p_1^2}{2} - \frac{(g(q_1) + a(q_2))^2}{2} - \alpha q_1, \quad q_2 = t$$

be a Hamiltonian linked to (13), and $\widehat{H} := H + p_2$ its autonomous completion.

Corollary 2.3 *If $X_{\widehat{H}}$ is integrable through rational or meromorphic first integrals then, along the integral curve $\Gamma = \{\boldsymbol{z}(t) = (q_1(t), t, \dot{q}_1(t), -H(t))\}$, the Galois group of the equation*

$$\ddot{\xi} = \left(g_{q_1}^2(q_1) + g_{q_1 q_1}(q_1)(g(q_1) + a(q_2))\right)\Big|_\Gamma \xi, \tag{14}$$

is virtually abelian. □

Now, keeping the above hypotheses for $g(x)$, $a(t)$, $\alpha = \alpha_0$, equation (13) and $x = q_1(t)$, let us define the Hamiltonian system

$$\widehat{H} = H + p_2, \quad H = \frac{p_1^2}{2} - (g(q_1) + a(q_2))p_1 - (\alpha + a_{q_2}(q_2))q_1, \quad \alpha \in \mathbb{C}. \tag{15}$$

It only takes the following simple calculation to prove that Hamiltonian H is indeed linked to (13) in the manner expected, i.e. as introduced in the beginning of Section 2. We have

$$\begin{array}{rcl} \ddot{x} & = & g_x(x)(g(x) + a(t)) + \alpha \\ & = & g_x(x)\dot{x} + g_x(x)g(x) + g_x(x)a(t) + \alpha + \dot{a}(t) - g_x(x)\dot{x} - \dot{a}(t) \\ & = & g_x(x)y + \alpha + \dot{a}(t) - \frac{d}{dt}(g(x) + a(t)), \end{array}$$

where $y = \dot{x} + g(x) + a(t)$, and requiring x and y to be conjugate variables implies the following system, equivalent to (13),

$$\begin{array}{rcccl} \dot{x} & = & y - (g(x) + a(t)) & = & H_y \\ \dot{y} & = & g_x(x)y + \alpha + \dot{a}(t) & = & -H_x. \end{array}$$

A straightforward parallel integration,

$$H = \frac{y^2}{2} - (g(x) + a(t))\, y + h_1(x,t) = h_2(y,t) - g(x)y - (\alpha + \dot{a}(t))x,$$

along with the definition of

$$h_1(x,t) = -(\alpha + \dot{a}(t))x, \quad h_2(y,t) = \frac{y^2}{2} - a(t)y,$$

as well as $(q_1, q_2, p_1) = (x, t, y)$, yields the autonomous Hamiltonian system introduced in (15).

Theorem 2.4 *If $X_{\widehat{H}}$ is integrable by means of rational or meromorphic first integrals, then, along $\Gamma = \{\boldsymbol{z}(t) = (q_1(t), t, p_1(t), -H(t))\}$, the Galois group of the equation*

$$\ddot{\xi} = \left(g_{q_1}^2(q_1) + p_1 g_{q_1 q_1}(q_1) - g_{q_2 q_1}(q_1)\right)\big|_\Gamma \xi, \tag{16}$$

is virtually abelian.

Proof. We may proceed as in the proof of Theorem 2.1. The Hamiltonian field $X_{\widehat{H}}$ is given by

$$X_{\widehat{H}} = \begin{pmatrix} p_1 - (g(q_1) + a(q_2)) \\ 1 \\ g_{q_1}(q_1)p_1 + (\alpha + a_{q_2}(q_2)) \\ a_{q_2}(q_2)p_1 + a_{q_2q_2}(q_2)q_1 \end{pmatrix}.$$

The variational equation VE_Γ along $\Gamma = \{(q_1(t), t, p_1(t), -H(t))\}$ is

$$\dot{\boldsymbol{\xi}} = \left.\begin{pmatrix} -\frac{\partial g(q_1)}{\partial q_1} & -\frac{\partial a(q_2)}{\partial q_2} & 1 & 0 \\ 0 & 0 & 0 & 0 \\ p_1\frac{\partial^2 g(q_1)}{\partial q_1^2} & \frac{\partial^2 a(q_2)}{\partial q_2^2} & \frac{\partial g(q_1)}{\partial q_1} & 0 \\ \frac{\partial^2 a(q_2)}{\partial q_2^2} & \left(p_1\frac{\partial^2 a(q_2)}{\partial q_2^2} + q_1\frac{\partial^3 a(q_2)}{\partial q_2^3}\right) & \frac{\partial a(q_2)}{\partial q_2} & 0 \end{pmatrix}\right|_\Gamma \boldsymbol{\xi}, \tag{17}$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3, \xi_4)^T$. $\xi_2 \equiv k \in \mathbb{C}$; in particular, $k = 0$ renders system (17) equal to the following:

$$\left\{\begin{array}{rcl} \dot{\xi}_1 & = & -g_{q_1}(q_1)|_\Gamma \xi_1 + \xi_3, \\ \dot{\xi}_2 & = & 0, \\ \dot{\xi}_3 & = & p_1 g_{q_1q_1}(q_1)|_\Gamma \xi_1 + -g_{q_1}(q_1)|_\Gamma \xi_3, \\ \dot{\xi}_4 & = & a_{q_2q_2}(q_2)|_\Gamma \xi_1 + a_{q_2}(q_2)|_\Gamma \xi_3, \end{array}\right. \tag{18}$$

NVE_Γ corresponding to

$$\frac{d}{dt}\begin{pmatrix} \xi_1 \\ \xi_3 \end{pmatrix} = \left.\begin{pmatrix} -g_{q_1}(q_1) & 1 \\ p_1 g_{q_1q_1}(q_1) & g_{q_1}(q_1) \end{pmatrix}\right|_\Gamma \begin{pmatrix} \xi_1 \\ \xi_3 \end{pmatrix}, \tag{19}$$

and solving (19) we obtain ξ_4. Using equations (2) and (3), system (19) is equivalent to equation (16), where $\xi = \xi_1$. Again in virtue of [20, Proposition 4.2] as in Theorem 2.1, the integrability of Hamiltonian $\widehat{H}$ by means of meromorphic or rational first integrals implies the virtual commutativity, of the Galois group of equation (16). ∎

Remark 3 We observe that if $\ddot{x} = f(x,t)$ with particular solution $x = x(t)$ is the same for Theorem 2.1, Corollary 2.3 and Theorem 2.4, then we obtain the same NVE_Γ (equations (9), (14) and (16) are equivalent), despite the fact that their respective linked Hamiltonian systems can be expressed differently.

3 Examples

In this Section, and in application of Theorems 2.1 and 2.4 as well as of Corollaries 2.2 and 2.3, we analyze the non-integrability of the Hamiltonian systems corresponding to the following differential equations:

1. Hill-Schrödinger equation: $\ddot{x} = k(t)x$,
2. Painlevé II equation: $\ddot{x} = 2x^3 + tx + \alpha$,
3. The differential equation: $\ddot{x} = -\frac{1}{4x^3} - \frac{t}{x^2} + \alpha$,
4. The Sitnikov problem: $\ddot{x} = -\frac{(1-e\cos t)x}{(x^2+r^2(t))^{\frac{3}{2}}}$, $r(t) = \frac{1-e\cos t}{2}$.

In order to analyze normal variational equations, a standard procedure is using Maple, and especially commands `dsolve` and `kovacicsols`. Whenever the command `kovacicsols` yields an output "`[ ]`", it means that the second-order linear differential equation being considered has no Liouvillian solutions, and thus its Galois group is virtually non-solvable. For equations of the form $\ddot{y} = ry$ with $r \in \mathbb{C}(x)$ the only virtually non-solvable group is $\mathrm{SL}_2(\mathbb{C})$. In some cases, moreover, `dsolve` makes it possible to obtain the solutions in terms of special functions such as *Airy functions*, *Bessel functions* and **hypergeometric functions**, among others ([2]). There is a number of second-order linear equations whose coefficients are not rational, and whose solutions Maple cannot find by means of the commands `dsolve` and `kovacicsols` alone; this problem, in some cases, can be solved by a previous algebrization procedure.

3.1 Hill-Schrödinger equation $\ddot{x} = k(t)x$

This example corresponds to Corollary 2.2. For $k > 0, \epsilon \gg 0$, $\mathcal{P}_n$ a polynomial of degree n with $\mathcal{P}_n(0) = 1$ and $k(t)$ given by

$$\begin{array}{ll} k(t) = k\mathrm{e}^{-\epsilon t}, & k(t) = k\mathcal{P}_n(\epsilon t), \\ k(t) = k(1+\sinh(\epsilon t)), & k(t) = k(1+\sin(\epsilon t)), \\ k(t) = k(1+\cosh(\epsilon t)), & k(t) = k(1+\cos(\epsilon t)). \end{array}$$

$X_{\widehat{H}}$ is non-integrable by means of rational first integrals.

The integrability of equation $\ddot{x} = k(t)x$ for these examples has been deeply analyzed in [3].

3.2 The Sitnikov problem

The *Sitnikov problem* is a symmetrically configured restricted three-body problem in which two primaries with equal masses move in ellipses of eccentricity e in a plane π_1, and an infinitesimal point mass moves along the line $\pi_1^\perp$. See [6], [15], [23], [25], [35], [36], [14] for more details. The motion of the infinitesimal point mass is given by the following differential equation

$$\ddot{z} + \frac{z}{\left(r^2\left(t\right) + z^2\right)^{3/2}} = 0, \tag{20}$$

where $z = z\left(t\right)$ is the distance from the infinitesimal mass point to the plane of the primaries and $r\left(t\right)$ is half the distance of the primaries,

$$r\left(t\right) = \frac{1 - e\cos E\left(t\right)}{2}.$$

where the *eccentric anomaly* $E\left(t\right)$ is the solution of the Kepler equation

$$E = t + e\sin E, \tag{21}$$

and e is the eccentricity of the ellipses described by the primaries. We will assume $0 \leq e \leq 1$ all though Subsections 3.2 and 3.3. The Hamiltonian linked to the system is

$$H = \frac{v^2}{2} - \frac{1}{\left(z^2\left(t\right) + r^2\left(t\right)\right)^{1/2}}, \tag{22}$$

provided v stands for $\dot{z}$ in the corresponding equations.

Since (22) cannot be solved in explicit form, attempts at a Hamiltonian formulation of (20), whether exact or approximate, require one of at least two options: looking for an exact Hamiltonian formulation by means of a change of variables, which we will do in the next paragraph, and searching an approximate Hamiltonian formulation, which will be done in Subsection 3.3.

Let us now find a Hamiltonian linked to 20. We may express $r\left(t\right)$ as

$$r\left(t\right) = \left(R \circ \varphi\right)\left(t\right) := \frac{1 - e^2}{2\left(1 + e\cos\varphi\left(t\right)\right)},$$

where φ, the *true anomaly*, is a solution of

$$\frac{d\varphi}{dt} = \frac{\left(1 + e\cos\varphi\right)^2}{\left(1 - e^2\right)^{3/2}} = \frac{\sqrt{1 - e^2}}{4R^2\left(\varphi\right)},$$

and we may follow the procedure introduced in [36] (see also [14]) by taking φ as the new independent variable and $x = \frac{z}{2r(\varphi)}$ as the new dependent variable. Writing t once again to denote φ, we have the following differential equation:

$$\ddot{x} = f\left(x, t\right) := -\frac{e\cos t + \left(\frac{1}{4} + x^2\right)^{-3/2}}{1 + e\cos t}x, \tag{23}$$

clearly amenable to the hypotheses in Theorem 2.1 and in the first paragraph of Section 2. Defining $q_1 = x$, $q_2 = t$ and $p_1 = \dot{x}$, the autonomous Hamiltonian system corresponding to (23) is given by

$$\widehat{H}_e = H_e + p_2 := \frac{p_1^2}{2} + \frac{eq_1^2 \cos q_2 - 4\left(1 + 4q_1^2\right)^{-1/2}}{2\left(1 + e\cos q_2\right)} + p_2, \tag{24}$$

always assuming $e \in [0, 1]$.

The *circular* Sitnikov problem $\widehat{H}_0$ is meromorphically integrable in the sense of Liouville-Arnold and can be solved using elliptic integrals. The non-integrability for $e = 1$ was first studied by means of straight Morales-Ramis theory in [24, §5] (see also [20, §5.3]); we will now extend the proof of meromorphic non-integrability therein to one for every $0 < e \leq 1$ by using Theorem 2.1.

The NVE_Γ derived from the Hamiltonian system given by (24) is

$$\ddot{\xi} = \left(\frac{e\left(4q_1^2 + 1\right)^{5/2} \cos q_2 - 64q_1^2 + 8}{\left(e\cos q_2 + 1\right)\left(4q_1^2 + 1\right)^{5/2}} \right) \xi.$$

Taking $q_1 \equiv 0$ we have a solution $\Gamma = \left\{ \boldsymbol{z}(t) = \left(0, t, 0, \frac{2}{1+e\cos t}\right) \right\}$ of $X_{\widehat{H}_e}$ along which NVE_Γ is given by

$$\ddot{\xi} = \left(\frac{e\cos t + 8}{e\cos t + 1} \right) \xi,$$

which is algebrizable, through the change of variable $\tau = \cos t$, into

$$\frac{d^2\xi}{d\tau^2} - \left(\frac{\tau}{1-\tau^2}\right)\frac{d\xi}{d\tau} - \left(\frac{e\tau + 8}{(e\tau+1)(1-\tau^2)}\right)\xi = 0. \tag{25}$$

This equation can be transformed into the differential equation

$$\frac{d^2\zeta}{d\tau^2} = \left(\frac{5e\tau^3 + 33\tau^2 - 2e\tau - 30}{(e\tau+1)4(1-\tau^2)^2}\right)\zeta, \tag{26}$$

by means of $\xi = \frac{\zeta}{\sqrt[4]{1-\tau^2}}$. Equations (25) and (26) are integrable in terms of Liouvillian solutions only if $e = 0$, whereas for $e \neq 0$ their solutions are given in terms of non-integrable Heun functions if $e \neq 1$ and non-integrable Hypergeometric functions if $e = 1$, i.e. their Galois groups are virtually non-solvable, hence virtually non-abelian; furthermore, they have a regular singularity at infinity, which by Theorem 1.3 implies the non-integrability of $X_{\widehat{H}}$ for $e \in (0, 1]$ by means of *meromorphic* first integrals. In particular, the Galois group of equation (26) is exactly $\mathrm{SL}_2(\mathbb{C})$.

3.2.1 Numerical results for the Sitnikov Problem

The author is indebted to Sergi Simon in what concerns the following subsection, including the figures shown at the end of the paper. Acknowledgments are also due to Carles Simó for further specific suggestions.

Let $\Sigma = \{\sin(q_2) = 0\}$. The six figures at the end of this paper show Poincaré sections of the flow with respect to Σ, projected on the (q_1, p_1) plane, for the Hamiltonian system X_{H_e} obtained from (24). As may be easily deduced from said Hamiltonian, all sections are symmetrical with respect to the q_1 and p_1 axes. Different amounts of initial conditions are used for the sake of clarity.

Figure 1 corresponds to $e = 0$. In keeping with what was said after (24), the whole subset of Σ transversal to the flow sheds concentric tori (ostensibly, the intersections of the invariant Liouville-Arnold tori with Σ), a typical sign of integrability; the tori shown are only a selection of those therein, as the actual area foliated by them is larger.

A number of these invariant tori break down upon the slightest increase in e, and in the ensuing figures the two most interesting features are those invariant sets (usually called *KAM tori*) whose intersection with Σ prevails in the form of Jordan curves, and the zones of chaotic behavior between them. Sparse zones of the section will account for chaotic zones as well, for $e > 0$. Figures 2 and 3 show two different close-up views for the Poincaré section corresponding to $e = 0.01$. The latter figure is actually a detail of the "island" of tori appearing at the right of the general section. For $e = 0.1$, Figures 4 and 5 are, respectively, a general view of the section and a close-up of one of the islands appearing at each side of the central area. As for $e = 0.4$, Figure 6 is an enlarged view of one of the two islands appearing at each side of a central area.

3.3 The approximate Sitnikov problem

As said in Subsection 3.2, we now consider an approximation of the Sitnikov problem; see [13] and [15] for more details. As opposed to *meromorphic* non-integrability, we will prove non-integrability by means of *rational* first integrals.

The fact that $\varphi(t) = t + O(e)$ yields $r(t) = \frac{1-e\cos t}{2} + O\left(e^2\right)$, and thus the Hamiltonian in (22) becomes

$$H = \frac{v^2}{2} - \frac{1}{\sqrt{z^2 + \frac{1}{4}}} - e\frac{\cos t}{\left(z^2 + \frac{1}{4}\right)^{3/2}} + O\left(e^2\right)$$

whenever $e \approx 0$. In particular, the first-order approximation of this asymptotic expansion in e yields the Hamiltonian

$$H = \frac{v^2}{2} - \frac{1}{\sqrt{z^2 + \frac{1}{4}}} - e\frac{\cos t}{\left(z^2 + \frac{1}{4}\right)^{3/2}} \tag{27}$$

Considering $q_1 = x$, $q_2 = t$ and $p_1 = v$; the autonomous Hamiltonian system corresponding to this equation is given by

$$\widehat{H} = H + p_2, \quad H = \frac{p_1^2}{2} - e\frac{2\cos q_2}{(4q_1^2+1)^{3/2}} - \frac{2}{\sqrt{4q_1^2+1}}, \tag{28}$$

corresponding to the Hamiltonian in Theorem 2.1:

$$f(x,t) = -\frac{8x}{(4x^2+1)^{\frac{3}{2}}} - e\frac{24x\cos t}{(4x^2+1)^{\frac{5}{2}}}.$$

The NVE_Γ for the Hamiltonian (28) is given by

$$\ddot{\xi} = \left(e\frac{24(16q_1^2-1)\cos q_2}{(4q_1^2+1)^{\frac{7}{2}}} + \frac{8(8q_1^2-1)}{(4q_1^2+1)^{\frac{5}{2}}} \right)\xi.$$

Its general solution may be expressed as:

$$\xi\,(t) = K_1 C\left(32, 48e, \frac{t}{2}\right) + K_2 S\left(32, 48e, \frac{t}{2}\right),$$

where the *Mathieu even* (resp. *odd*) *function* $C\,(a,q,t)$ (resp. $S\,(a,q,t)$) is defined as the even (resp. odd) solution to $\ddot{y} + (a - 2q\cos{(2t)})\,y = 0$ ([2, Ch. 20]).

Taking $q_1(t) = 0$ we can see that $\boldsymbol{z}(t) = (0, t, 0, 2e\cos t + 2)$; hence, defining $\boldsymbol{z}(t) = (0, t, 0, 2e\cos t + 2)$ and $\Gamma = \{\boldsymbol{z}(t)\}$, the operator linked to NVE_Γ is given by

$$\ddot{\xi} = (-24e\cos t - 8)\,\xi,$$

which is algebrizable (see Theorem 1.2) through the change of variables $\tau = \cos t$ into

$$\frac{d^2\xi}{d\tau^2} - \left(\frac{\tau}{1-\tau^2}\right)\frac{d\xi}{d\tau} + \left(\frac{24e\tau+8}{1-\tau^2}\right)\xi = 0. \tag{29}$$

Now, this equation can be transformed in the differential equation

$$\frac{d^2\zeta}{d\tau^2} = \left(\frac{96e\tau^3+31\tau^2-96e\tau-34}{4(1-\tau^2)^2}\right)\zeta, \quad \xi = \frac{\zeta}{\sqrt[4]{1-\tau^2}}. \tag{30}$$

Equations (29) and (30) are integrable in terms of Liouvillian solutions only if $e = 0$, since for $e \neq 0$ their solutions are given in terms of non-integrable Mathieu functions, hence their Galois group are virtually non-solvable and thus virtually non-abelian; furthermore, they have an irregular singularity at infinity, implying rational non-integrability for the Hamiltonian field $X_{\widehat{H}}$ with $\alpha = 1$ in virtue of Theorem 1.3. In particular, the Galois group of equation (30) is exactly $\mathrm{SL}_2\,(\mathbb{C})$.

Equations (29) and (30) have been deeply analyzed in [3] using the Hamiltonian change of variables $\tau = e^{it}$, obtaining the same result presented here.

3.4 Painlevé II equation: $\ddot{x} = 2x^3 + tx + \alpha$

Defining $q_1 = x$, $q_2 = t$, $p_1 = y$ and $\alpha \in \mathbb{C}$, the autonomous Hamiltonian system corresponding to this equation can given by any of the following three functions:

$$\widehat{H} = H + p_2, \quad H = \frac{p_1^2}{2} - \frac{q_1^4}{2} - q_2\frac{q_1^2}{2} - \alpha q_1, \tag{31}$$

$$\widehat{H} = H + p_2, \quad H = \frac{p_1^2}{2} - \frac{1}{2}\left(q_1^2 + \frac{q_2}{2}\right)^2 - \alpha q_1, \tag{32}$$

$$\widehat{H} = H + p_2, \quad H = \frac{p_1^2}{2} - \left(q_1^2 + \frac{q_2}{2}\right)p_1 - \left(\alpha + \frac{1}{2}\right)q_1, \tag{33}$$

where the equations (31), (32) and (33) correspond to the Hamiltonian of Theorem 2.1 ($f(x,t) = 2x^3 + tx + \alpha$), Corollary 2.3 ($g(x) = x^2$ and $a(t) = t/2$) and Theorem 2.4 ($g(x) = x^2$ and $a(t) = t/2$) respectively. The Hamiltonian system for Painlevé II, studied in [21, 27], corresponds precisely to Hamiltonian (33). The NVE_Γ for these Hamiltonians is given by

$$\ddot{\xi} = \left(6q_1^2 + q_2\right)\xi.$$

Taking $\alpha = 0$ and $q_1(t) = 0$ we have particular solutions $\boldsymbol{z}(t) = (0,t,0,0)$, $\boldsymbol{z}(t) = \left(0,t,0,t^2/8\right)$ and $\boldsymbol{z}(t) = \left(0,t,t/2,t^2/8\right)$, respectively, for the Hamiltonians (31), (32) and (33); hence NVE_Γ is given by $\ddot{\xi} = t\xi$, the so-called *Airy equation* ([2, §10.4.1]), which has an irregular singularity at infinity and is not integrable through Liouvillian solutions, i.e. its Galois group is $\mathrm{SL}_2(\mathbb{C})$, not virtually abelian; thus, by Theorem 1.3, the Hamiltonian field $X_{\widehat{H}}$ with $\alpha = 0$ is not integrable through rational first integrals.

Now, for $\alpha = 1$ and $q_1(t) = -1/t$, the integral curve $\boldsymbol{z}(t)$ is given by

$$\left(-\frac{1}{t}, t, \frac{1}{t^2}, -\frac{1}{2t}\right), \quad \left(-\frac{1}{t}, t, \frac{1}{t^2}, -\frac{1}{2t} + \frac{t^2}{8}\right) \text{ and } \left(-\frac{1}{t}, t, \frac{2}{t^2} + \frac{t}{2}, -\frac{1}{t} + \frac{t^2}{8}\right),$$

respectively for the Hamiltonians (31), (32) and (33), so that NVE_Γ is given by

$$\ddot{\xi} = \left(\frac{6}{t^2} + t\right)\xi, \quad \Gamma = \{\boldsymbol{z}(t)\},$$

whose general solution is

$$\xi(t) = \sqrt{t}\left[K_1 I_{-5/3}\left(\frac{2t^{3/2}}{3}\right) + K_2 I_{5/3}\left(\frac{2t^{3/2}}{3}\right)\right],$$

$I_\alpha = 2^{-\alpha}t^\alpha\left(\frac{1}{\Gamma(1+\alpha)} + \frac{t^2}{2^2\Gamma(2+\alpha)} + O\left(t^4\right)\right)$ being, for each α, the *modified Bessel function of the first kind*, i.e. the solution to $t^2\ddot{y} + t\dot{y} - \left(t^2 + \alpha^2\right)y = 0$ ([2, §9.6]).

The normal variational equation has an irregular singularity at infinity and is not integrable through Liouvillian functions because its solutions are given in term of non-integrable Bessel functions (see [21, 27]), i.e. its Galois group is $\mathrm{SL}_2(\mathbb{C})$ which is not virtually abelian; again by the Morales-Ramis Theorem 1.3, the Hamiltonian field $X_{\widehat{H}}$ with $\alpha = 1$ is not integrable through rational first integrals.

3.5 The differential equation: $\ddot{x} = -\frac{1}{4x^3} - \frac{t}{x^2} + \alpha$

Considering $q_1 = x$, $q_2 = t$, $p_1 = y$ and $\alpha \in \mathbb{C}$; the autonomous Hamiltonian systems corresponding to this equation are given by

$$\widehat{H} = H + p_2, \quad H = \frac{p_1^2}{2} - \frac{1}{8q_1^2} - \frac{q_2}{q_1} - \alpha q_1, \tag{34}$$

$$\widehat{H} = H + p_2, \quad H = \frac{p_1^2}{2} - \frac{1}{2}\left(\frac{1}{2q_1} + 2q_2\right)^2 - \alpha q_1, \tag{35}$$

$$\widehat{H} = H + p_2, \quad H = \frac{p_1^2}{2} + \left(\frac{1}{2q_1} + 2q_2\right) p_1 - (\alpha + 2)\, q_1. \tag{36}$$

(34), (35) and (36) correspond to Theorem 2.1 ($f(x,t) = -\frac{1}{4x^3} - \frac{t}{x^2} + \alpha$), Corollary 2.3 ($g(x) = -\frac{1}{2x}$ and $a(t) = -2t$) and Theorem 2.4 ($g(x) = -\frac{1}{2x}$ and $a(t) = -2t$) respectively. The NVE_Γ for all three is given by

$$\ddot{\xi} = \left(\frac{3}{4q_1^4} + \frac{2q_2}{q_1^3}\right)\xi.$$

Now, for $\alpha = 1$ and $q_1(t) = \sqrt{t}$, the integral curve $\boldsymbol{z}(t)$ is given by

$$\left(\sqrt{t}, t, \frac{1}{2\sqrt{t}}, 2\sqrt{t}\right), \quad \left(\sqrt{t}, t, \frac{1}{2\sqrt{t}}, 2t^2 + 2\sqrt{t}\right) \text{ and } \left(\sqrt{t}, t, -2t, 2t^2\right),$$

respectively for the Hamiltonians (34), (35) and (36), rendering NVE_Γ equal to

$$\ddot{\xi} = \left(\frac{3}{4t^2} + \frac{2}{\sqrt{t}}\right)\xi, , \tag{37}$$

having a solution

$$\begin{aligned} \xi_1 &= -\frac{3t^{3/2}}{2}\, {}_0F_1\left(;\frac{7}{3};\frac{8t^{3/2}}{9}\right) \\ &= -\frac{3}{2}t^{3/2} - \frac{4}{7}t^3 - \frac{8}{105}t^{9/2} + O\left(t^6\right), \end{aligned}$$

${}_0F_1(;a;t) = \lim_{q\to\infty} {}_1F_1\left(q;a;\frac{t}{q}\right) = \sum_{n=0}^{\infty} \frac{t^n}{(a)_n n!}$ being the *confluent hypergeometric limit function* ([2, Ch. 13]), and an independent new solution $\xi_2 = \xi_1 \int \xi_1^{-2}$, satisfying

$$\xi_2 = \frac{1}{3\sqrt{t}} - \frac{8t}{9} - \frac{16}{27}t^{5/2} + O\left(t^4\right).$$

As is the case for the rest of normal variational operators appearing in this paper, our knowledge of the exponents around 0 of a fundamental set of solutions (in this case, ξ_1 and ξ_2), coupled with the basic result on factorization obtained in [8, Th. 8 (Ch. 5)] (see also [8, Criterion 1]) would suffice to prove non-integrability. Here, however, we will keep our restriction to Theorems 2.1 and 2.4 and Corollary 2.3.

(37) is algebrizable (Theorem 1.2), through the change of variables $\tau = \sqrt{t}$ into

$$\frac{d^2\xi}{d\tau^2} - \left(\frac{1}{\tau}\right)\frac{d\xi}{d\tau} - \left(\frac{8\tau^3+3}{\tau^2}\right)\xi = 0, \tag{38}$$

now, this equation can be transformed in the differential equation

$$\frac{d^2\zeta}{d\tau^2} = \left(\frac{32\tau^3+15}{4\tau^2}\right)\zeta, \quad \xi = \zeta\sqrt{\tau}. \tag{39}$$

Equations (38) and (39) have an irregular singularity at $t = \infty$ and are not integrable through Liouvillian solutions due to the presence of Bessel functions, i.e. their Galois group are virtually non-solvable, therefore virtually non-abelian, Theorem 1.3 once again settling rational non-integrability for $\alpha = 1$. In particular, the Galois group of equation (39) is exactly $\mathrm{SL}_2(\mathbb{C})$.

4 Final Remarks: Open Questions and Future Work

This paper is the starting point of a project in which the author is involved. The following questions arose during our work:

- In [21, 27] it was proven that the autonomous Hamiltonian system related to Painlevé II is non-integrable for every $\alpha \in \mathbb{Z}$. Is this also true for equation (13)?
- Does the integrability of equation (13) for arbitrary $\alpha \in \mathbb{Z}$ depend on the choice of $g(x)$ and $a(t)$?
- Assuming the above question has an affirmative answer, in what manner can the choice and form of $g(x)$ and $a(t)$ assure non-integrability for every $\alpha \in \mathbb{Z}$? and for every $\alpha \in \mathbb{C}$?
- Is it possible to find transversal sections of the flow, and thus Poincaré maps, for either $\widehat{H}$ or the algebraized equation, even in the absence of non-trivial numerical monodromies? Do Stokes multipliers contribute to the answer in a significant manner?

Among our next goals, the analysis of the following items is due further immediate research:

- the application of Morales-Ramis theory to higher variational equations of *NAHS*;
- differential equations in the form $\ddot{x} = f(x, \dot{x}, t)$;
- the rest of Painlevé equations;
- the theoretical aspects of *NAHS* such as their geometry and the feasibility of an analogue to Liouville-Arnold theory;

- the non-integrability of *NAHS* with two and a half degrees of freedom;
- specific examples of *NAHS* related to control theory, as well as other related to Celestial Mechanics, such as Restricted Three- and Four-Body Problems and Hénon-Heiles systems ([4], [11]).
- the exact relation, perhaps causal, between separatrix splitting ([22]) and non-integrability, whether rational or meromorphic.

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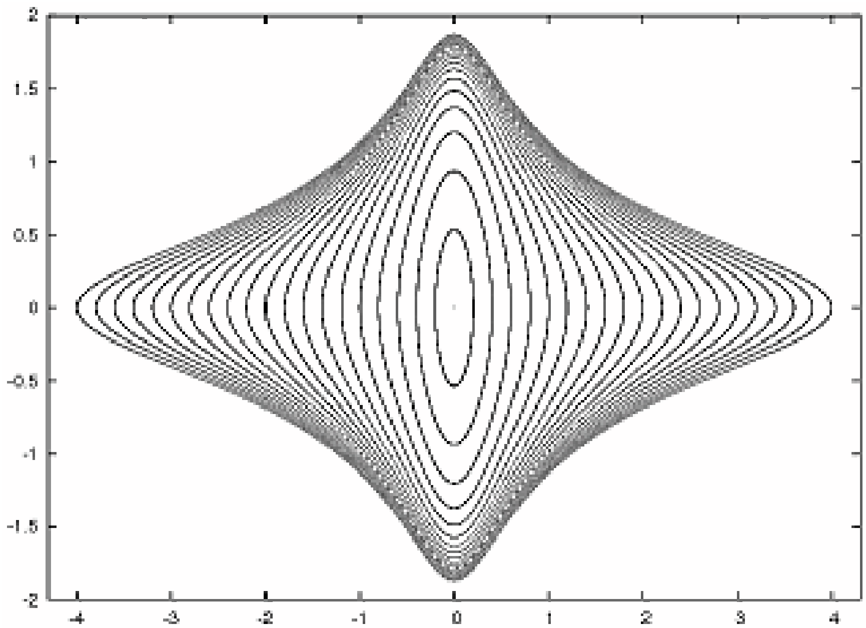

Figure 1: Poincaré section $\sin q_2 = 0$, $e = 0$ (Figure: Sergi Simon)

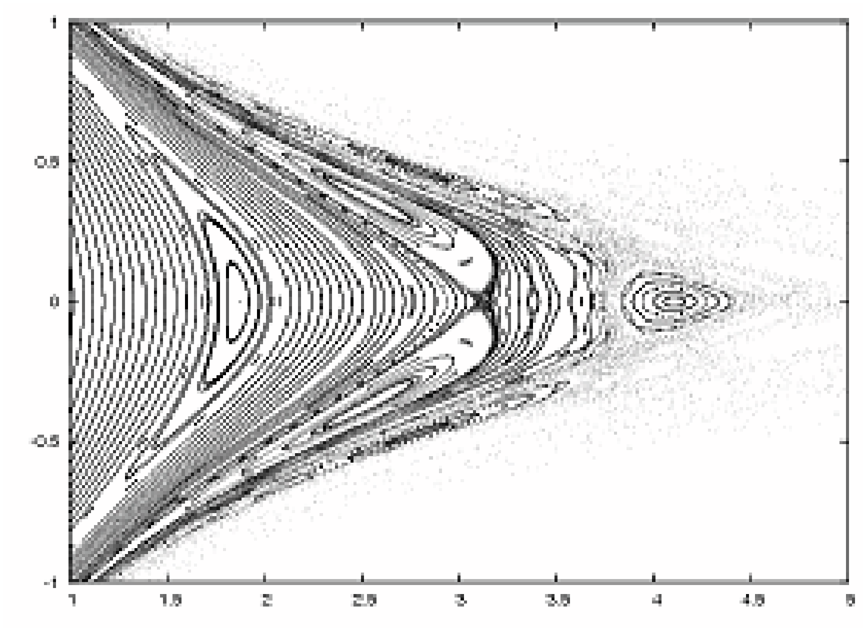

Figure 2: Poincaré section $\sin q_2 = 0$, $e = 0.01$ (Figure: Sergi Simon)

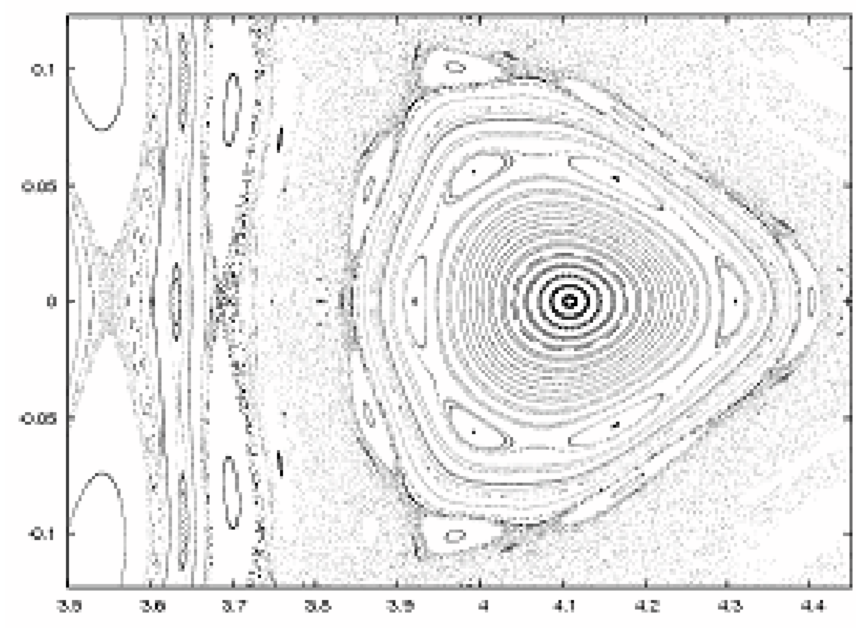

Figure 3: Poincaré section for $e = 0.01$ (Figure: Sergi Simon)

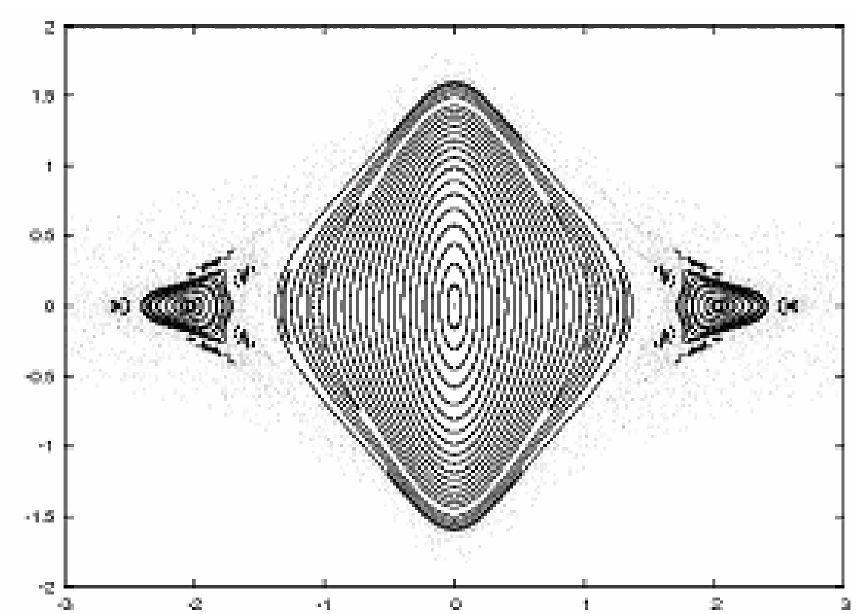

Figure 4: Poincaré section $\sin q_2 = 0$, $e = 0.1$ (Figure: Sergi Simon)

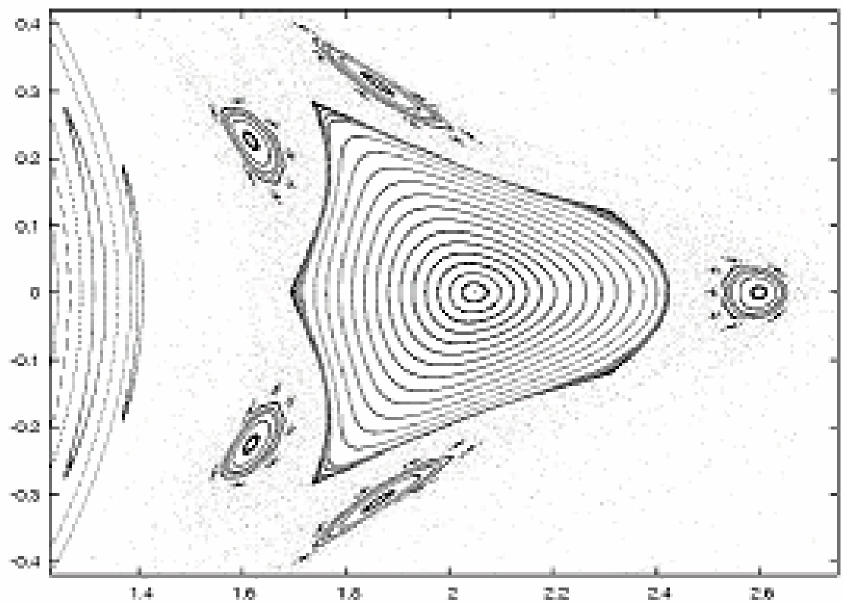

Figure 5: Poincaré section for $e = 0.1$ (Figure: Sergi Simon)

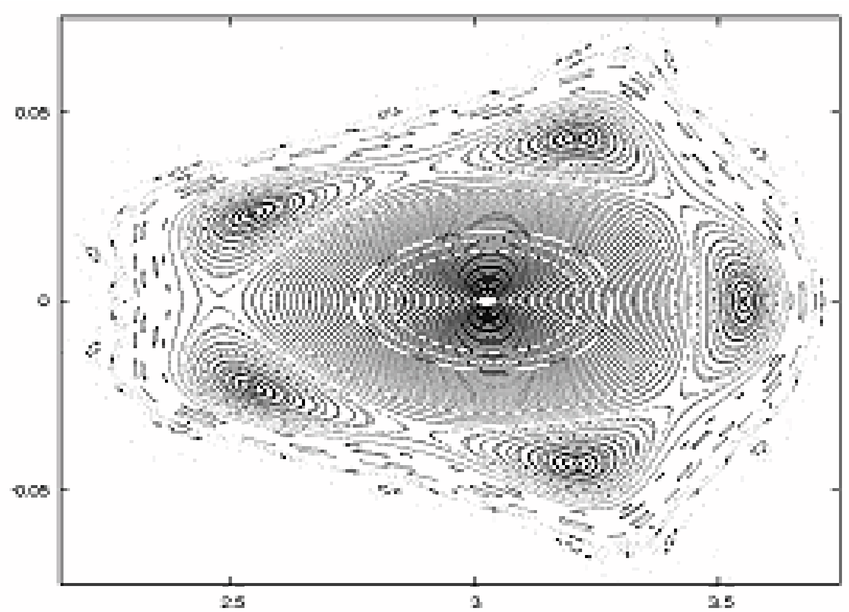

Figure 6: Poincaré section, $e = 0.4$ (Figure: Sergi Simon)